MATHEMATICS

MEASURE-THEORETICAL INVESTIGATIONS CONCERNING CONTINUED FRACTIONS

BY

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I. *Introduction*

A. Let θ be a *real* number $(0<\theta<1)$.

We expand θ in the continued fraction:

$$\theta=\{a_1, a_2, a_3, \ldots\}=\cfrac{1}{a_1+\cfrac{1}{a_2+\cfrac{1}{a_3+\ldots}}} \tag{1}$$

with integer partial quotients $a_n \geqq 1$.

For a *rational* θ we have uniquely:

$$\theta=\{a_1, a_2, a_3, \ldots, a_N\} \; (N \geqq 1;\, a_N \geqq 2). \tag{2}$$

In this case we shall call θ a *rational of order* N and we define:

$$\begin{cases} \theta_0=\theta,\, \theta_1=\{a_2, a_3, \ldots, a_N\},\, \theta_2=\{a_3, a_4, \ldots, a_N\}, \ldots, \\ \theta_{N-1}=\{a_N\},\, \theta_{N+\nu}=0 \; (\nu=0, 1, 2, \ldots). \end{cases} \tag{2a}$$

For an *irrational* θ we define:

$$\theta_0=\theta \text{ and } \theta_n=\{a_{n+1}, a_{n+2}, a_{n+3}, \ldots\} \; (n \geqq 1). \tag{2b}$$

It is clear that by (2a, b) θ_n $(n=0, 1, 2, \ldots)$ is defined as a function of θ on $0<\theta<1$. If we define:

$$\theta_n=0 \; (n=0, 1, 2, \ldots) \text{ for } \theta=0 \text{ and } \theta=1 \tag{2c}$$

by (2a, b, c) θ_n as a function of θ is defined on $0 \leqq \theta \leqq 1$.

For $n \geqq 0$ and $\theta_n \neq 0$, because of the relation

$$\theta_n=\frac{1}{a_{n+1}+\theta_{n+1}} \tag{3}$$

we have:

$$\theta_{n+1}=\frac{1}{\theta_n}-\left[\frac{1}{\theta_n}\right] \tag{4}$$

and:

$$a_{n+1}=\left[\frac{1}{\theta_n}\right]. \tag{4a}$$

B. Let $f(x)$ be a real function on $0 \leqq x \leqq 1$. Let D be a division of the interval $\langle 0, 1\rangle$ with dividing points $x_0(=0)$, $x_1, x_2, \ldots, x_P(=1)$. Further put

$$\sigma_p = \sup f(x) - \inf f(x)$$

where both, sup and inf, are to be extended over the closed interval $x_p \leqslant x \leqslant x_{p+1}$.

Then

$$\varphi(D) = \sum_{p=0}^{P-1} \sigma_p (x_{p+1} - x_p)$$

denotes the "oscillation of $f(x)$ for the division D".

In our thesis [4] we proved the following lemma:

Lemma I: *Let D_k be the division of $\langle 0, 1\rangle$ with dividing points*

$$h_0^k\,(=0),\ h_1^k\left(=\frac{1}{2^k}\right),\ h_2^k\left(=\frac{2}{2^k}\right),\ \ldots,\ h_{2^k}^k(=1).$$

Let $f(x)$ be a real function on $0 \leqq x \leqq 1$, such that for the sequence $\{D_k\}$ of divisions D_k of $\langle 0, 1\rangle$ the corresponding oscillations satisfy

$$\sum_{k=1}^{\infty} \varphi(D_k) < \infty. \tag{5}$$

Then for any couple of integers $M \geqslant 0$, $N \geqslant 1$ we have:

$$I \stackrel{\text{def}}{=} \int_0^1 \left\{ \sum_{n=M+1}^{M+N} \left(f(\theta_n) - \int_0^1 \frac{f(x)}{(1+x)\log 2}\, dx \right) \right\}^2 d\theta \leqq K_1 N,$$

K_1 being a constant depending only on $f(x)$.

It is not difficult to prove that a function $f(x)$ which satisfies the condition (5), also satisfies the condition (5) for each sequence $\{D_k'\}$ of arbitrary divisions D_k' such that the distance between two successive dividing points of D_k' does not exceed $1/2^k$.

C. In our thesis we gave several applications of lemma I, which we shall not repeat here. New investigations taught us that the conditions imposed upon $f(x)$ in lemma I can be chosen somewhat weaker. In this note we shall give a proof of the lemma with the new conditions (in II) and give some new applications (in III). All applications given in the thesis concern bounded functions $f(x)$. Lemma II enables us to consider also other functions, e.g. $\log x$ $(0 < x \leqslant 1)$. One of the new applications delivers a refinement of a well-known result, due to A. Khintchine [2].

After the appearance of the thesis our attention was drawn to the work of C. Ryll-Nardzewski [3], who investigated the numbers θ_n belonging to the continued fraction θ by means of the individual ergodic theorem. His methods and results are different from ours. Whereas he proves the relation

$$\frac{1}{N}\sum_{n=1}^{N} f(\theta_n) \to \int_0^1 \frac{f(x)}{(1+x)\log 2}\,dx$$

for all $f(x) \in L$, we, for much narrower classes of functions $f(x)$, derive estimates for the difference

$$\frac{1}{N}\sum_{n=1}^{N} f(\theta_n) - \int_0^1 \frac{f(x)}{(1+x)\log 2}\,dx.$$

For the meaning of the main notations used in the sequel we refer to our thesis [4].

II. Let $f(x)$ be a real function defined on the interval $0<x\leqq 1$. Let D_k be a division of $(0, 1\rangle$ with the dividing points

$$h_1^k\left(=\frac{1}{2^k}\right),\ h_2^k\left(=\frac{2}{2^k}\right),\ h_3^k\left(=\frac{3}{2^k}\right), \ldots, h_{2^k}^k(=1).$$

Further put

$$\sigma_p = \sup f(x) - \inf f(x) \quad (1 \leqq p \leqq 2^k-1)$$

where both, sup and inf, are to be extended over the closed interval $x_p \leqslant x \leqslant x_{p+1}$.

Then:

$$\varphi(D_k) = \sum_{p=1}^{2^k-1} \sigma_p\,(x_{p+1}-x_p) \tag{6}$$

we shall call the oscillation for the division D_k of the function $f(x)$ defined on $(0, 1\rangle$.

Now we can prove the following lemma:

Lemma II: *Let $f(x)$ be a non-negative (or non-positive) function defined on the interval $0<x\leqq 1$, such that for two positive constants β and K*

$$|f(x)| \leqq \frac{K}{x^{\frac{1}{2}-\beta}}.$$

Further let for the sequence $\{D_k\}$ of divisions D_k of the interval $(0, 1\rangle$ the corresponding oscillations (6) satisfy

$$\sum_{k=1}^{\infty} \varphi(D_k) < \infty.$$

Then for any couple of integers $M \geqslant 0$, $N \geqslant 1$ we have:

$$I \stackrel{\text{def}}{=} \int_0^1 \left\{ \sum_{n=M+1}^{M+N} \left(f(\theta_n) - \int_0^1 \frac{f(x)}{(1+x)\log 2}\,dx \right) \right\}^2 d\theta \leqq K_2 \cdot N$$

K_2 being a constant only depending on $f(x)$.

Remark: Our proof involves the existence of I as a Lebesgue-integral.

Proof of lemma II.

In this article the numbers $\delta_1, \delta_2, \ldots$, are real numbers with $|\delta_k| \leqq 1$ $(k=1, 2, \ldots)$.

A. Consider the case that $f(x) \geqslant 0$.

1. We note that from the conditions of lemma II it follows that

$$\int_0^1 f(x)dx, \int_0^1 f^2(x)dx, \int_0^1 \frac{f(x)}{(1+x)\log 2}\,dx \text{ and } \int_0^1 \frac{f^2(x)}{(1+x)\log 2}\,dx$$

exist as (im)proper Riemann integrals.

2. Put:

$$\int_0^1 \frac{f(x)}{(1+x)\log 2}\,dx = P. \tag{7}$$

Then:

$$\int_0^1 f(x)\,dx \leqq 2P. \tag{8}$$

Now we shall prove that $f(\theta_n) \in L^2$. Then obviously

$$I = \sum_{n=M+1}^{M+N} \sum_{m=M+1}^{M+N} \left\{ \int_0^1 f(\theta_n)\, f(\theta_m) d\theta - P \int_0^1 f(\theta_n) d\theta - P \int_0^1 f(\theta_m) d\theta + P^2 \right\} \tag{9}$$

exists as a Lebesgue integral. In order to study $\int_0^1 f(\theta_n)d\theta$, $\int_0^1 f^2(\theta_n)d\theta$ and $\int_0^1 f(\theta_n)\, f(\theta_m)d\theta$ we define a sequence of functions $f_q(x)$ $(q=1, 2, 3, \ldots)$ as follows:

$$\begin{aligned} f_q(x) &\equiv f(x) \quad \text{for } 1/q \leqslant x \leqslant 1 \\ f_q(x) &= 0 \qquad \text{for } 0 \leqslant x < 1/q \ (q=1, 2, 3, \ldots). \end{aligned}$$

From the conditions imposed upon $f(x)$ it follows that, for every positive integer q, $f_q(x)$ is R-integrable on $0 \leqslant x \leqslant 1$ and therefore (by [4]; Lemma X) that $f_q(\theta_n(\theta))$ is R-integrable on $0 \leqslant \theta \leqslant 1$. Defining

$$m(x) = mV\{\theta;\, \theta_n \leqslant x\}$$

we get (as we proved in [4]; p. 34–35)

$$\int_0^1 f_q(\theta_n)d\theta = \int_0^1 f_q(x)\,\frac{dm(x)}{dx}\,dx = \int_{1/q}^1 f(x)\,\frac{dm(x)}{dx}\,dx.$$

Now $f_1(\theta_n), f_2(\theta_n), f_3(\theta_n), \ldots$ is a non-decreasing sequence of non-negative L-integrable functions with the limit $f(\theta_n)$ for all θ $(0 \leqslant \theta \leqslant 1)$ except a set with measure zero. According to a convergence theorem for monotone sequences (cf. e.g. E. C. Titchmarsh: The theory of functions; 10.82), as

$$\lim_{q\to\infty} \int_0^1 f_q(\theta_n)d\theta = \lim_{q\to\infty} \int_{1/q}^1 f(x)\,\frac{dm(x)}{dx}\,dx = \int_0^1 f(x)\,\frac{dm(x)}{dx}\,dx$$

the function $f(\theta_n)$ is L-integrable on $0 \leqslant \theta \leqslant 1$ and (Cf. [4]; (33))

$$(10)\quad \begin{cases} \int_0^1 f(\theta_n)d\theta = \int_0^1 f(x)\,\dfrac{dm(x)}{dx}\,dx = \int_0^1 \dfrac{f(x)}{(1+x)\log 2}\,dx + \int_0^1 f(x)\delta_1 A e^{-\alpha\sqrt{n}}\,dx = \\ \qquad\qquad = P + \delta_2 2APe^{-\alpha\sqrt{n}}. \end{cases}$$

Therefore:

$$(11)\quad -\sum_{n=M+1}^{M+N}\ \sum_{m=M+1}^{M+N} P\int_0^1 f(\theta_n)d\theta = -\sum_{n=M+1}^{M+N}\ \sum_{m=M+1}^{M+N} P^2 + 2\delta_3 AP^2QN$$

with $Q = \sum_{n=1}^{\infty} e^{-\alpha\sqrt{n}}$.

In the same way we get:

$$(12)\quad -\sum_{n=M+1}^{M+N}\ \sum_{m=M+1}^{M+N} P\int_0^1 f(\theta_m)d\theta = -\sum_{n=M+1}^{M+N}\ \sum_{m=M+1}^{M+N} P^2 + 2\delta_4 AP^2QN.$$

3. If we put:

$$(13)\qquad \int_0^1 \frac{f^2(x)}{(1+x)\log 2}\,dx = \overline{P}$$

we have:

$$(14)\qquad \int_0^1 f^2(x)dx \leqq 2\overline{P}.$$

Now, as from the conditions imposed upon $f(x)$ it follows that the functions $f_1(\theta_n), f_2(\theta_n), f_3(\theta_n), \ldots$ are L^2-integrable, in a similar manner as we proved (10), we find that $f(\theta_n)$ is L^2-integrable on $0 \leqslant \theta \leqslant 1$ and

$$(15)\qquad \int_0^1 f^2(\theta_n)d\theta = \int_0^1 f^2(x)\,\frac{dm(x)}{dx}\,dx = \overline{P} + 2\delta_5 A\overline{P}e^{-\alpha\sqrt{n}}.$$

Hence:

$$(16)\qquad \sum_{n=M+1}^{M+N} \int_0^1 f^2(\theta_n)d\theta = \overline{P}N + 2\delta_6 A\overline{P}Q.$$

Further we have:

$$(17)\quad \begin{cases} \sum\limits_{n=M+1}^{M+N}\ \sum\limits_{m=M+1}^{M+N} \int_0^1 f(\theta_n)\,f(\theta_m)d\theta = \sum\limits_{n=M+1}^{M+N} \int_0^1 f^2(\theta_n)d\theta + \\ \qquad\qquad + 2\sum\limits_{n=M+1}^{M+N-1}\ \sum\limits_{\substack{m=M+2\\ m>n}}^{M+N} \int_0^1 f(\theta_n)\,f(\theta_m)d\theta. \end{cases}$$

In order to estimate

$$\sum_{n=M+1}^{M+N-1}\ \sum_{\substack{m=M+2\\ m>n}}^{M+N} \int_0^1 f(\theta_n)\,f(\theta_m)d\theta$$

we define:

$$k = [\tfrac{1}{2}(m-n)] + 1 \tag{18}$$

and denote by h_t^k and $l_{(D)}^k$ the numbers defined in ([4]; p. 24).

Further we define:

$$\varepsilon_k = h_1^k. \tag{19}$$

4. We shall first give an estimate for $\int\limits_{\substack{0\\ \theta_n > \varepsilon_k}}^{1} f(\theta_n)\, f(\theta_m) d\theta$.

If we define:

$$V_{k,t} = V\{\theta\,;\, h_t^k < \theta_n \leqslant h_{t+1}^k\}$$

we get:

$$\int\limits_{\substack{0\\ \theta_n > \varepsilon_k}}^{1} f(\theta_n)\, f(\theta_m) d\theta = \lim_{q\to\infty} \int\limits_{\substack{0\\ \theta_n > \varepsilon_k}}^{1} f(\theta_n)\, f_q(\theta_m) d\theta = \lim_{q\to\infty} \sum_{t=0}^{T-1} \xi_{k,t} \int\limits_{V_{k,t}} f_q(\theta_m) d\theta$$

with

$$\inf f(x) \leqslant \xi_{k,t} \leqslant \sup f(x)$$

where both, inf and sup, are to be extended over the closed interval $h_t^k \leqslant x \leqslant h_{t+1}^k$.

If we now define: $\bar{m}(x) = mV\{\theta\,;\, \theta_n \in V_{k,t};\, \theta_m \leqq x\}$ in a similar manner as ([4]; (38)–(43)) we get:

$$\tag{20} \left\{ \begin{aligned} & \lim_{q\to\infty} \sum_{t=0}^{T-1} \xi_{k,t} \int\limits_{V_{k,t}} f_q(\theta_m) d\theta = \lim_{q\to\infty} \sum_{t=0}^{T-1} \xi_{k,t} \int\limits_{0}^{1} f_q(x) \frac{d\bar{m}(x)}{dx}\, dx = \\ & = \lim_{q\to\infty} \sum_{t=0}^{T-1} \xi_{k,t}\, mV_{k,t} \left(\int\limits_0^1 \frac{f_q(x)}{(1+x)\log 2} dx + \int\limits_0^1 f_q(x) \delta_7 A e^{-\alpha\sqrt{m-n-k-1}} dx \right) = \\ & = (P + 2\delta_8 A P e^{-\alpha\sqrt{m-n-k-1}}) \left(\sum_{t=0}^{T-1} \xi_{k,t}\, mV_{k,t} \right) = \\ & \qquad\qquad = (P + 2\delta_8 A P e^{-\alpha\sqrt{m-n-k-1}}) \left(\int\limits_{\substack{0\\ \theta_n > \varepsilon_k}}^{1} f(\theta_n) d\theta + \varrho_k \right) \end{aligned} \right.$$

(with $|\varrho_k| \leqq 2\varphi(k)$).

Now we have:

$$\tag{21} \left\{ \begin{aligned} \int\limits_{\substack{0\\ \theta_n > \varepsilon_k}}^{1} f(\theta_n) d\theta &= \int\limits_{\varepsilon_k}^{1} f(x) \frac{dm(x)}{dx} dx = \int\limits_{\varepsilon_k}^{1} \frac{f(x)}{(1+x)\log 2} dx + \int\limits_{\varepsilon_k}^{1} f(x) \delta_9 A e^{-\alpha\sqrt{n}}\, dx \\ &= P + 2\delta_{10} A P e^{-\alpha\sqrt{n}} - \int\limits_0^{\varepsilon_k} \frac{f(x)}{(1+x)\log 2}\, dx. \end{aligned} \right.$$

As $\varepsilon_k = 1/2^k$ and as $f(x) \leqq K x^{-\frac{1}{2}+\beta}$ we find:

$$\int\limits_0^{\varepsilon_k} \frac{f(x)}{(1+x)\log 2}\, dx \leqq \frac{3}{2} \int\limits_0^{\varepsilon_k} f(x) dx \leqq \frac{3K}{2^{\frac{1}{2}k}}. \tag{22}$$

Combination of (20), (21) and (22) gives:

$$(23)\quad \left\{\begin{aligned} &\int\limits_{\substack{0\\ \theta_n>\varepsilon_k}}^{1} f(\theta_n)\, f(\theta_n) d\theta = \\ &\qquad = (P+2\delta_{11}APe^{-\alpha\sqrt{m-n-k-1}})\left(P+2\delta_{12}APe^{-\alpha\sqrt{n}}+\varrho_k+\frac{3\delta_{13}K}{2^{\frac{1}{2}k}}\right).\end{aligned}\right.$$

As $\sum\limits_{k=1}^{\infty} 1/2^{\frac{1}{2}k}$ is a convergent series, in a similar manner as we deduced ([4]; (47)) here we get:

$$(24)\quad \sum_{n=M+1}^{M+N-1}\ \sum_{\substack{m=M+2\\ m>n}}^{M+N}\ \int\limits_{\substack{0\\ \theta_n>\varepsilon_k}}^{1} f(\theta_n)\, f(\theta_m) d\theta = \sum_{n=M+1}^{M+N-1}\ \sum_{\substack{m=M+2\\ m>n}}^{M+N} (P^2) + \delta_{14}K_3N$$

K_3 being a constant only depending on $f(x)$.

5. Finally we have to estimate:

$$\sum_{n=M+1}^{M+N-1}\ \sum_{\substack{m=M+2\\ m>n}}^{M+N}\ \int\limits_{\substack{0\\ \theta_n\leqq\varepsilon_k}}^{1} f(\theta_n)\, f(\theta_m) d\theta.$$

As $\varepsilon_k = 1/2^k$ and as $f(x) \leqq Kx^{-\frac{1}{2}}$ we get:

$$(25)\quad \left\{\begin{aligned} &\int\limits_{\theta_n\leqq 1/2^k} f(\theta_n)\, f(\theta_m) d\theta = \sum_{\nu=2^k}^{\infty}\ \int\limits_{1/\nu+1<\theta_n\leqq 1/\nu} f(\theta_n)\, f(\theta_m) d\theta \leqslant \\ &\qquad\qquad \leqslant \sum_{\nu=2^k}^{\infty} K(\nu+1)^{\frac{1}{2}}\cdot \int\limits_{1/\nu+1<\theta_n\leqq 1/\nu} f(\theta_m) d\theta.\end{aligned}\right.$$

As $1/\nu$ and $1/\nu+1$ both are rationals of order 1 and as

$$mV\left\{\theta;\frac{1}{\nu+1}<\theta_n\leqq\frac{1}{\nu}\right\}\leqq\frac{2}{\nu(\nu+1)}\quad \text{(cf. [4]; (8a))}$$

defining $\bar{m}(x) = mV\{\theta;\ 1/\nu+1<\theta_n\leqslant 1/\nu;\ \theta_m\leqslant x\}$ we get:

$$(26)\quad \left\{\begin{aligned} &\int\limits_{1/\nu+1<\theta_n\leqq 1/\nu} f(\theta_m) d\theta = \int\limits_0^1 f(x)\frac{d\bar{m}(x)}{dx}\,dx = mV\left\{\theta;\frac{1}{\nu+1}<\theta_n\leqq\frac{1}{\nu}\right\}\times \\ &\qquad \left\{\int\limits_0^1 \frac{f(x)}{(1+x)\log 2}\,dx + \int\limits_0^1 f(x)\,\delta_{15}Ae^{-\alpha\sqrt{m-n-1}}\,dx\right\} \leqq \frac{2(P+2PA)}{\nu(\nu+1)}.\end{aligned}\right.$$

By (25) and (26) we have:

$$(27)\quad \left\{\begin{aligned} &\int\limits_{\theta_n\leqq\varepsilon_k} f(\theta_n)\, f(\theta_m) d\theta \leqq \sum_{\nu=2^k}^{\infty}\frac{2K(P+2PA)}{\nu(\nu+1)^{\frac{1}{2}}} \leqq 2K(P+2PA)\int\limits_{2^{\frac{1}{2}k}}^{\infty}\frac{1}{y^{1\frac{1}{2}}}\,dy \leqq \\ &\qquad\qquad \leqq \frac{4K(P+2PA)}{2^{\frac{1}{2}k}}.\end{aligned}\right.$$

As $\sum\limits_{k=1}^{\infty} 1/2^{\frac{1}{2}k}$ converges we get by (27)

$$(28)\quad \sum_{n=M+1}^{M+N-1}\ \sum_{\substack{m=M+2\\ m>n}}^{M+N}\ \int\limits_{\theta_n\leqq\varepsilon_k} f(\theta_n)\, f(\theta_m) d\theta \leqq K_4N,$$

K_4 being a constant only depending on $f(x)$.

6. Combination of the results (9), (11), (12), (16), (17), (24) and (28) proves lemma II.

B. Consider the case that $f(x) \leqslant 0$.

Put

$$g(x) \equiv - f(x);$$

as $g(x) \geqq 0$ lemma II is valid for the function $g(x)$.

As $\left\{ \sum_{n=M+1}^{M+N} f(\theta_n) - \int_0^1 \frac{f(x)}{(1+x) \log 2} dx \right\}^2 = \left\{ \sum_{n=M+1}^{M+N} g(\theta_n) - \int_0^1 \frac{g(x)}{(1+x) \log 2} dx \right\}^2$

lemma II is valid for the function $f(x)$ too.

III. *Applications*

Applying a theorem of I. S. GÀL and J. F. KOKSMA [1] (cf. [4]; lemma VII) on lemma II we get the following

Theorem I: *Let $f(x)$ be a real function defined on* $(0,1\rangle$, *satisfying the conditions of lemma II.*

Then for almost all θ we have:

$$\sum_{n=1}^{N} f(\theta_n) - N \int_0^1 \frac{f(x)}{(1+x) \log 2} dx = o(N^{\frac{1}{2}} \log^{\frac{3+\varepsilon}{2}} N). \tag{29}$$

We give some applications.

a) Put $f(x) = \frac{1}{x^{\frac{1}{2}-\beta}}$. Then for almost all θ we have:

$$\sum_{n=1}^{N} \frac{1}{\theta_n^{\frac{1}{2}-\beta}} - N \int_0^1 \frac{1}{x^{\frac{1}{2}-\beta}(1+x) \log 2} dx = o(N^{\frac{1}{2}} \log^{\frac{3+\varepsilon}{2}} N). \tag{30}$$

b) Put $f(x) = \log x$. Then for almost all θ we have:

$$\sum_{n=1}^{N} \log (\theta_n) - N \int_0^1 \frac{\log x}{(1+x) \log 2} dx = o(N^{\frac{1}{2}} \log^{\frac{3+\varepsilon}{2}} N). \tag{31}$$

c) By the relation (4a) the number a_{n+1} is, for an irrational θ, completely defined by the number θ_n. Now, let be given a positive function $f(p)$ of the positive integer p such that $f(p) \leqq Kp^{\frac{1}{2}-\beta}$ (K and β being positive constants).

As $f(a_{n+1}) = f\left(\left[\frac{1}{\theta_n}\right]\right)$ and as $\int_0^1 \frac{f\left(\left[\frac{1}{x}\right]\right)}{(1+x) \log 2} dx =$

$$= \sum_{p=1}^{\infty} \int_{1/p+1}^{1/p} \frac{f(p)}{(1+x) \log 2} dx = \sum_{p=1}^{\infty} f(p) \frac{\log \left(1 + \frac{1}{p(p+2)}\right)}{\log 2}$$

we have for almost all θ:

$$\sum_{n=1}^{N} f(a_n) - N \sum_{p=1}^{\infty} f(p) \frac{\log\left(1+\dfrac{1}{p(p+2)}\right)}{\log 2} = o(N^{\frac{1}{2}} \log^{\frac{3+\varepsilon}{2}} N). \tag{32}$$

Putting $f(p) = \log p$ we get:

$$\sum_{n=1}^{N} \log a_n - N \sum_{p=1}^{\infty} \log\left(1+\frac{1}{p(p+2)}\right)^{\frac{\log p}{\log 2}} = o(N^{\frac{1}{2}} \log^{\frac{3+\varepsilon}{2}} N) \tag{33}$$

for almost all θ.

The last result we can write otherwise as follows:

$$\sqrt[N]{a_1, a_2, \ldots, a_N} = \prod_{p=1}^{\infty} \left(1+\frac{1}{p(p+2)}\right)^{\frac{\log p}{\log 2}} + o(N^{-\frac{1}{2}} \log^{\frac{3+\varepsilon}{2}} N) \tag{34}$$

for almost all θ.

(34) is a refinement of a theorem of A. Khintchine (cf. [2]; p. 376).

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